SRS PU COLLEGE, CHITRADURGA

(in coordination with Narayana Group of Institutions, Hyderabad)

II PU MATHEMATICS ANNUAL EXAM - MARCH 2020

SUBJECT: MATHEMATICS (35) **Max. Marks-100**

INSTRUCTIONS **Time: 3.15 Hrs**

This question paper has 4 parts, all parts are compulsory.

- **Part-A carries 10 marks. Each question carries one mark.**
- **Part-B carries 20 marks. Each question carries two marks.**
- **Part-C carries 30 marks. Each question carries three marks.**
- **Part-D carries 30 marks. Each question carries five marks.**
- **Part-D carries 10 marks. Each question carries ten marks.**

PART-A

I. Answer all the questions. Each question carries one mark

1. Let * be the binary operation on N given by $a * b = L.C.M$ of a and b. Find $5 * 7$.

   Ans: $5 * 7 = L.C.M$ of 5 and 7 =35

2. Write the range of the function $y = \sec^{-1} x$.

   Ans: $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

3. If a matrix has 5 elements, what are the possible orders it can have?

   Ans: 1×5 and 5×1

4. Find the value of x for which

   $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

   Ans: $x^2 - 36 = 36 - 36$

   $x^2 - 36 = 0$

   $x^2 = 36$

   $x = \pm 6$

5. If $y = \tan(\sqrt{x}), find \frac{dy}{dx}$.

   Ans: $\frac{dy}{dx} = sec^2(\sqrt{x})\frac{1}{2\sqrt{x}}$

6. Find $\int(2x^2 + e^x)dx$.

   Ans: $2\frac{x^3}{3} + e^x + C$

7. Define negative of a vector.

   Ans: A vector whose magnitude is the same as that of a given vector (say $\overline{AB}$) but direction is opposite to that of it is called the negative of the given vector

   i.e., $\overrightarrow{BA} = -\overrightarrow{AB}$

8. If a line makes angles 90^0, 135^0 and 45^0 with the X, Y and $Z - axes$ respectively, find its direction cosines.

   Ans: Direction cosines are : $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

9. **Define optimal solution in Linear programming problem.**

Ans: Any point in the feasible region that gives the optimal value (maximum of minimum) of the objective function is called an optimal solution.

10. **If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$ find $P(A \cap B)$ if A and B are independent events.**

Ans: $P(A \cap B) = P(A)P(B) = \frac{3}{5} \times \frac{1}{5} = \frac{3}{25}$

PART-B

II. Answer any ten of the following questions.

11. **If $f: R \to R$ and $g: R \to R$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$ find gof and fog.**

Ans: $gof\ (x) = g[f(x)] = g[\cos x] = 3\cos^2 x$.

$fog(x) = f[g(x)] = f(3x^2) = \cos(3x^2)$

12. **Prove that $\cot^{-1}(-x) = \pi - \cot^{-1} x, \forall x \in R$.**

Ans: Let $cot^{-1}(-x) = \theta$

$cot\,\theta = -x$

$x = -cot\theta$

$x = cot(\pi - \theta)$

$cot^{-1}x = \pi - \theta$

$\theta = \pi - cot^{-1}x$

$\therefore cot^{-1}(-x) = \pi - cot^{-1}x$.

13. **Find the value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$.**

Ans: $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{3\pi}{5}\right)\right]$

$= \sin^{-1}\left[\sin\frac{2\pi}{5}\right]$

$= \frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

14. **Find the area of the triangle whose vertices are $(-2, -3)$, $(3, 2)$ and $(-1, -8)$ using determinant method.**

Ans: Area of the triangle = $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$

$= \frac{1}{2}[-2\,(2 + 8 + 3(3 + 1) + 1(-24 + 2)]$

$\Delta = 15\ sq.units.$

15. **Find $\frac{dy}{dx}$, if $\sin^2 x + \cos^2 y = 1$.**

Ans: $\sin^2 x + \cos^2 y = 1$

Differentiate with respect to x,

$2\,sinx\cos x + 2\cos y\,(-siny)\frac{dy}{dx} = 0$

$\sin 2x - \sin 2y \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$

16. If $y = x^x$, find $\frac{dy}{dx}$.

Ans: $y = x^x$

Take log on both sides

$\log y = \log x^x$

$\log y = x \log x$

Differentiate with respect to x

$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x} + \log x$

$\frac{dy}{dx} = x^x(1 + \log x)$

17. Find the interval in which the function f given by $f(x) = x^2 - 4x + 6$

Ans: $f(x) = x^2 - 4x + 6$

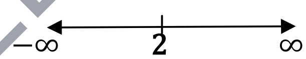
$f'(x) = 2x - 4 = 2(x - 2)$

$f'(x) = 0$

$2(x - 2) = 0$

$x = 2$

$\therefore$ *the intervals are*.



$(-\infty, 2)$ and $(2, \infty)$

when $x \in (-\infty, 2)$, $f'(x) = 2(x - 2) = -ve < 0$

when $x \in (2, \infty)$, $f'(x) = +ve > 0$

$\therefore$ $f(x)$ *is stritly decreasing in* $(-\infty, 2)$ and strictly increasing in $(2, \infty)$

18. Find $\int \cot x \log(\sin x) dx$.

Ans: $\int \cot x \log(\sin x)\, dx$

$= \int t\, dt = \frac{t^2}{2} + C$ | put t= $\log \sin x$

$= \frac{(\log(\sin x))^2}{2} + C$ | $dt = \cot x\, dx$

19. Find $\int x \sec^2 x\, dx$.

Ans: $\int x \sec^2 x\, dx$ $\quad u = x , dv = \sec^2 x$

$= uv - \int v\, du$

$= x \tan x - \int \tan x\, dx$

$= x \tan x - \log|\sec x| + C$

20. Find the order and degree (if defined of the differential equation

$$\left[\frac{d^2y}{dx^2}\right]^3 + \left[\frac{dy}{dx}\right]^2 + \sin\left[\frac{dy}{x}\right] + 1 = 0$$

Ans: order =2

degree = not defined.

21. Find the projection of the vector $\vec{a} = \hat{\imath} + 3\hat{\jmath} + 7\hat{k}$ on the vector $\vec{b} = 7\hat{\imath} - \hat{\jmath} + 8\hat{k}$.

Ans: projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$= \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$$

22. Fine the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{b} = 2\hat{\imath} - 7\hat{\jmath} + \hat{k}$.

Ans: Are of a parallelogram $= |\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = 20\,\hat{\imath} + 5\hat{\jmath} - 5\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{450} = 15\sqrt{2}$$

$= 15\sqrt{2}$ sq. units

23. Find the equation of the plane with intercepts 2,3 and 4 on the X, Y and Z- axes respectively.

Ans: Let the equations of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$,

given $a = 2$, $b = 3$, $and\ c = 4$

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$$

$\Rightarrow 6x + 4y + 3z = 12$

24. A random variable X has the following probability distribution.

X	0	1	2	3	4
P(X)	0.1	K	2K	2K	K

Ans:

We know that $\sum_{i=1}^{n} p_i = 1$

$\therefore P(o) + P(1) + P(2) + P(3) + P(4) = 1$

$0.1 + k + 2k + 2k + k = 1$

$6k = 1 - 0.1 = 0.9$

$k = \frac{0.9}{6} \Rightarrow k = 0.15$

PART-C

III. Answer any ten of the following questions.

25. Show that the relation R defined in the set A of all triangles as $R = \{(T_1, T_2): T_1$ is similar to $T_2\}$ is an equivalence relation.

$R = \{(T_1, T_2): T_1$ triangle is similar to T_2 triangle$\}$.

i) R is reflexive because $T_1 R T_1$

ii) R is symmetric because

$T_1 R T_2 \Rightarrow T_1$ and T_2 are similar $\Rightarrow T_2$ and T_1 are similar

$\Rightarrow T_2 R T_1 \Rightarrow$ R is symmetric

iii) R is transitive because $T_1 R T_2$ and $T_2 R T_3$

$\Rightarrow T_1$ is similar to T_2 and T_2 is similar to T_3

$\Rightarrow T_1$ is similar to $T_3 \Rightarrow T_1 R T_3 \Rightarrow$ R is transitive.

26. Prove that $2^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right)$.

$2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \tan^{-1}\left(\frac{2\left(\frac{1}{2}\right)}{1-\frac{1}{4}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \tan^{-1}\left(\frac{1}{\frac{3}{4}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \tan^{-1}\left[\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}}\right]$

$\Rightarrow \tan^{-1}\left[\frac{28+3}{21-4}\right] \Rightarrow \tan^{-1}\left[\frac{31}{17}\right]$ RHS

27. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then show that $F(x)F(y) = F(x+y)$.

We have, $F(x)=\begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore\ F(y)=\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore F(x)F(y)=\begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} \cos x\cos y-\sin x\sin y & -\cos x\sin y-\sin x\cos y & 0 \\ \sin x\cos y+\cos x\sin y & -\sin x\sin y+\cos x\cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}=F(x+y)$$

$\therefore\ F(x)F(y)=F(x+y)$

28. If $x = 2at^2, y = at^4$ then find $\frac{dy}{dx}$.

Given

$x = 2at^2$; $y = at^4$

$\frac{dx}{dt} = 2a(2t)$ $\quad$ $\frac{dy}{dt} = 4at^3$

$\frac{dx}{dt} = 4at$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2$$

$$\boxed{\frac{dy}{dx} = t^2}$$

29. Verify mean value theorem for the function $f(x) = x^2 - 4x - 3, x \in [1, 4]$.

Sol, Given $f(x) = x^2 - 4x - 3$ is a polynomial function

We know that Every polynomial function is Continuous in $[1,4]$ and differentiable on $(1,4)$

$f(x) = x^2 - 4x - 3$

$f'(x) = 2x - 4$

$f'(c) = 2c - 4$

$f(x) = x^2 - 4x - 3$

$f(1) = (1)^2 - 4(1) - 3$

$f(1) = -6$

$f(4) = (4)^2 - 4(4) - 3$

$f(4) = -3$

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$2c - 4 = \frac{-3 + 6}{4 - 1}$

$2c - 4 = 1$

$2c = 5$

$c = 2.5 \in (1,4)$

Hence Mean Value theorem is Verified

30. Use differential to approximate $\sqrt{36.6}$.

Sol $f(x) = \sqrt{x}$; $36.6 = 36 + 0.6$

Here $x = 36$; $\delta x = 0.6$

$f(x + \delta x) = f(x) + f'(x)\,\delta x$

$\sqrt{36.6} = \sqrt{x} + \frac{1}{2\sqrt{x}}\,\delta x$

$\sqrt{36.6} = \sqrt{36} + \frac{1}{2\sqrt{36}} \cdot 0.6$

$= 6 + \frac{1}{12_2} \times \frac{6}{10}$

$= 6 + (0.5 \times 0.1)$

$= 6 + 0.05$

$= 6.05$

$\therefore \sqrt{36.6} = 6.05$

31. Find $\int \frac{(x-3)e^x}{(x-1)^3} dx$.

$= \int e^x \left[\frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3} \right] dx$

$= \int e^x \left[\frac{1}{(x-1)^2} + \frac{-2}{(x-1)^3} \right] dx$

$\frac{1}{(x-1)^2} \to f(x)$, $\frac{-2}{(x-1)^3} \to f'(x)$

$= \frac{e^x}{(x-1)^2} + c$

$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$

32. **Evaluate :** $\int_0^{\pi/2} \cos^2 x\, dx$.

Sol $\int_0^{\pi/2} \cos^2 x \, dx = \int_0^{\pi/2} \frac{1+\cos 2x}{2} dx$

$\Rightarrow \frac{1}{2} \int_0^{\pi/2} (1+\cos 2x)\, dx$

$\Rightarrow \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi/2}$

$\Rightarrow \frac{1}{2} \left[\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - (0-0) \right]$

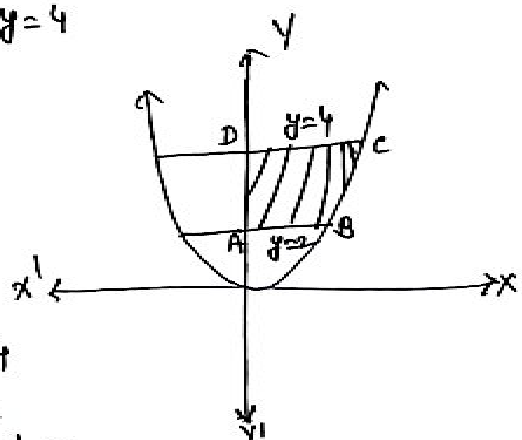
$\Rightarrow \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$

$$\boxed{\int_0^{\pi/2} \cos^2 x\, dx = \frac{\pi}{4}}$$

33. **Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the $y - axes$ in the first quadrant.**

Sol Given $x^2 = 4y$ | $y = 2$; $y = 4$

$x = 2\sqrt{y}$



Area of the region bounded by the Curve $A = 2\int_2^4 \sqrt{y}\, dy$

$A = 2 \cdot \frac{2}{3} \left[y^{3/2} \right]_2^4$

$A = \frac{4}{3} \left[4^{3/2} - 2^{3/2} \right]$

$A = \frac{4}{3} \left[8 - 2\sqrt{2} \right]$

$$\boxed{A = \frac{8}{3} \left[4 - \sqrt{2} \right]}$$

34. **Find the equation of a curve passing through the point $(-2, 3)$, given that the slope of the tangent to the curve tr any point (x, y) is $\frac{2x}{y^2}$.**

$$\int y^2\, dy = \int 2x\, dx$$

$$\frac{y^3}{3} = 2\frac{x^2}{2} + c \longrightarrow (1)$$

It passes through $(-2, 3)$

$$\frac{(3)^3}{3} = (-2)^2 + c$$

$$9 = 4 + c$$

$$\boxed{c = 5}$$

Substituting 'c' in (1)

$$\frac{y^3}{3} = x^2 + 5$$ which is the required equation

35. **Find a unit vector perpendicular to each of the vector $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ where $a = 3\hat{\imath} + 2\hat{\jmath} + 2\hat{\jmath} + 2\hat{k}$ and $b = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$.**

Sol) Let $\vec{c} = \vec{a} + \vec{b}$

$\vec{c} = 4\hat{i} + 4\hat{j} + 0\hat{k}$

$\vec{d} = \vec{a} - \vec{b}$

$\vec{d} = 2\hat{i} + 0\hat{j} + 4\hat{k}$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$= \hat{i}(16-0) - \hat{j}(16-0) + \hat{k}(0-8)$$

$$= 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$$= 8(2\hat{i} - 2\hat{j} - \hat{k})$$

$$|\vec{c} \times \vec{d}| = 8\sqrt{4+4+1} = 8 \times 3$$

Unit vector $\perp$r to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is

$$= \frac{(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})}{|(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})|}$$

$$= \frac{8(2\hat{i} - 2\hat{j} - \hat{k})}{8 \times 3}$$

$$= \left(\frac{2\hat{i} - 2\hat{j} - \hat{k}}{3}\right)$$

36. **Find x such that the four points $A(3,2,1), B(4,x,5), c(4,2,-2)$ and $D(6,5,-1)$ are coplanar.**

let $\overline{OA} = 3\hat{i} + 2\hat{j} + \hat{k}$

$\overline{OB} = 4\hat{i} + x\hat{j} + 5\hat{k}$

$\overline{OC} = 4\hat{i} + 2\hat{j} - 2\hat{k}$

$\overline{OD} = 6\hat{i} + 5\hat{j} - \hat{k}$

$\overline{AB} = \overline{OB} - \overline{OA}$

$= \hat{i} + (x-2)\hat{j} + 4\hat{k}$

$\overline{AC} = \overline{OC} - \overline{OA}$

$= \hat{i} + 0\hat{j} - 3\hat{k}$

$\overline{AD} = \overline{OD} - \overline{OA}$

$= 3\hat{i} + 3\hat{j} - 2\hat{k}$

given that the four points are Coplanar

ie, $[\overline{AB} \quad \overline{AC} \quad \overline{AD}] = 0$

$$\begin{vmatrix} 1 & (x-2) & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$1(0+9) - (x-2)(-2+9) + 4(3-0) = 0$

$9 - (x-2)7 + 12 = 0$

$9 - 7x + 14 + 12 = 0$

$7x = 35$

$\boxed{x = 5}$

$\therefore$ The Value of x is 5

37. **Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point $(2,2,1)$.**

Sol Given planes $\pi_1 \equiv 3x - y + 2z - 4 = 0$

$\pi_2 \equiv x + y + z - 2 = 0$

Equation of the plane passing through the intersection of the planes is $\pi_1 + \lambda \pi_2 = 0$

$(3x - y + 2z - 4) + \lambda (x + y + z - 2) = 0 \longrightarrow (1)$

It passes through the point $(2,2,1)$

$[3(2) - 2 + 2(1) - 4] + \lambda [2 + 2 + 1 - 2] = 0$

$$2 + 3\lambda = 0$$

$$3\lambda = -2$$

$$\boxed{\lambda = -\frac{2}{3}}$$

Substituting 'λ' in ①

$$(3x - y + 2z - 4) - \frac{2}{3}(x + y + z - 2) = 0$$

$$9x - 3y + 6z - 12 - 2x - 2y - 2z + 4 = 0$$

$$\boxed{7x - 5y + 4z - 8 = 0}$$

38. A man is known to speak truth 3 out of 4 times. He throws a dice and reports that it is a six. Find the probability that it is actually a six.

$P(S_1)$ = probability that six occurs $= \frac{1}{6}$

$P(S_2)$ = probability that six doesnot occurs $= \frac{5}{6}$.

E be the event that the man reports that six occurs in the throwing of the die.

$P(E/S_1) = \frac{3}{4}$ $\quad$ $P(E/S_2) = 1 - \frac{3}{4} = \frac{1}{4}$.

$$P(S_1/E) = \frac{P(S_1) \cdot P(E/S_1)}{P(S_1) \cdot P(E/S_1) + P(S_2) \cdot P(E/S_2)}$$

$$= \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4}} = \frac{1}{8} * \frac{24}{8} = \frac{3}{8}.$$

Hence required probability $= 3/8$

PART-D

IV. Answer any six of the following questions.

39. Show that the function $f: R \to R$ given $f(x) = 4x + 3$ is invertible. Find the inverse of

ANS:

Given function is $f(x) = 4x + 3, \forall x \in R$

Let $x_1, x_2 \in N$ then $f(x_1) = f(x_2)$

$\Rightarrow 4x_1 + 3 = 4x_2 + 3$

$\Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one

Let $y \in R$ then $y = f(x)$

$\Rightarrow y = 4x + 3 \Rightarrow y - 3 = 4x$

$\Rightarrow x = \frac{y-3}{4} \in N, \forall y \in R$

$\therefore$ Corresponding to every $y \in R$ there exists $\frac{y-3}{4} \in R$ such that $f\left(\frac{y-3}{4}\right) = y$

$\therefore$ f is onto.

Hence f is both one-one and onto f is invertible f^{-1} exists.

To find: f^{-1};

Let $y = f(x)\ (\because x = f^{-1}(y))$

$\Rightarrow y = 4x + 3 \Rightarrow x = \frac{y-3}{4} \Rightarrow f^{-1}(y) = \frac{y-3}{4}$

or $f^{-1}(x) = \frac{x-3}{4}$

40. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ then compute $(A + B)$ and $(B - C)$. Also verify that $A + (B - C) = (A + B) - C$.

$$A + B = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}, \quad B - C \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\therefore A + (B - C) = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \quad \ldots(1)$$

and

$$(A + B) - C = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \quad \ldots(2)$$

From (1) and (2) we get

$A + (B - C) = (A + B) - C$

41. **Solve the system of linear equations by matrix method**

$2x + 3y + 3z = 5$

$x - 2y + z = -4$

$3x - y - 2z = 3$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

Now, $|A| = 2(4+1) - 3(-2-3) + 3(-1+6)$

$= 10 + 15 + 15 = 40 \neq 0$

$\Rightarrow$ Given system has unique solution.

We have, $A^{-1} = \frac{adj\ A}{|A|}$

To find adj A:

$A_{11} = (4+1) = 5$, $A_{12} = -(-2-3)$
$= 5$, $A_{13} = (-1+6) = 5$

$A_{21} = -(-6+3) = 3$, $A_{22} = (-4-9)$
$= -13$, $A_{23} = -(-2-9) = 11$

$A_{31} = (3+6) = 9$, $A_{32} = -(2-3) = 1$,
$A_{33} = (-4-3) = -7$

$$\therefore \text{adj } A = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Since $AX = B$

$\Rightarrow X = A^{-1}B$

$$\Rightarrow X = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow x = 1, y = 2, z = -1$$

42. **If $y = (tan^{-1}x)^2$, show that $(x^2+1)^2 y^2 + 2x(x^2+1)y_1 = 2$.**

$y = (tan^{-1}x)^2$

$\frac{dy}{dx} = 2tan^{-1}x\ \frac{1}{1+x^2}$

Multiply by $1 + x^2$

$1 + x^2\ \frac{dy}{dx} = 2tan^{-1}x$

$1 + x^2\ \frac{d^2y}{dx^2} + \frac{dy}{dx}(2x) = 2\frac{1}{1+x^2}$

Multiply by $1 + x^2$

$(1+x^2)^2\frac{d^2y}{dx^2} + \frac{dy}{dx}(2x)(1+x^2) = 2$

$(1+x^2)^2\frac{d^2y}{dx^2} + 2x\ 1 + x^2\ \frac{dy}{dx} = 2$

43. **Sand is puring a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?**

Let, r, h and V be the radius, height and volume of the cone at any time t.

Given $\frac{dV}{dt} = 12cm^3/s$

$h = \frac{1}{6}r$ i.e. $r = 6h$

We know that $V = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi(6h)^2 h$$
$$V = 12\pi h^3$$

Diff. w.r.to t

$$\frac{dv}{dt} = 12\pi 3h^2 \frac{dh}{dt}$$
$$12 = 12\pi 3h^2 \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{1}{3\pi h^2}$$
$$\frac{dh}{dt} = \frac{1}{48\pi} cm/s.$$

44. Find the integral of $\frac{1}{x^2+a^2}$ $w.r.t\ x$ and hence evaluate $\int \frac{1}{x^2+2x+2}dx$.

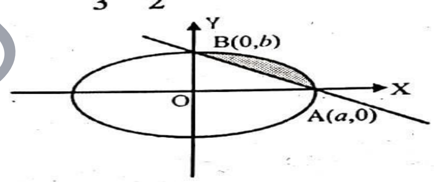
Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta d\theta$

$$\therefore\ I = \int \frac{1}{x^2+a^2}dx = \int \frac{1}{a^2\tan^2\theta + a^2} a\sec^2\theta\, d\theta$$
$$= \int \frac{a\sec^2\theta}{a^2(\tan^2\theta+1)}d\theta = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$$
$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
$$I = \int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1}$$
$$= \tan^{-1}(x+1) + C$$

45. Using the method of integration, find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$.

The ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ ...(1)

and the line $\frac{x}{3} + \frac{y}{2} = 1$ ...(2)



can be drawn, as shown in the given figure.
Then, we have to find area of the shaded region.
Required area is shown shaded

$$= \int_0^3 \frac{2}{3}\sqrt{9-x^2}\, dx - \int_0^3 \frac{2}{3}(3-x)\, dx$$
$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3 - \frac{2}{3}\left[3x - \frac{x^2}{2}\right]_0^3$$
$$= \frac{2}{3}\left[0 + \frac{9}{2}\frac{\pi}{2} - 0 - 0\right] - \frac{2}{3}\left[9 - \frac{9}{2} - 0 + 0\right]$$
$$= \frac{3\pi}{2} - 3 \text{ square unit.}$$

46. Find the general solution of the differential equation $x\frac{dy}{dx}+2y=x^2 logx$.

Given differential equation

$$\frac{dy}{dx}+\frac{2}{x}y=x\log x$$

Here $P=\frac{2}{x}$, $Q=x\log x$

$$\therefore I\cdot F=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$$

$\therefore$ Required solution is

$$yx^2=\int(x\log x)x^2dx+C=\int x^3\log x\,dx+C$$

$$=(\log x)\frac{x^4}{4}-\int\frac{x^4}{4}\cdot\frac{1}{x}dx+C$$

$$=\frac{x^4}{4}(\log x)-\frac{1}{4}\frac{x^4}{4}+C=\frac{x^4}{4}\log x-\frac{x^4}{16}+C$$

47. Derive the equation of a line in space passing through a given point and parallel to a given vector in both vector and Cartesian form.

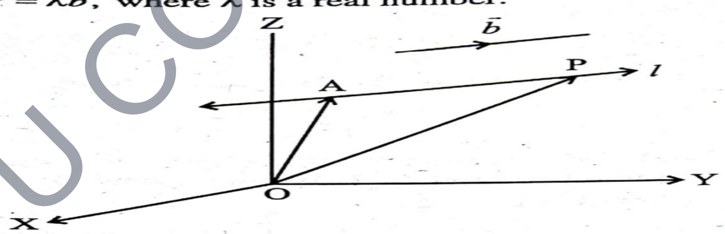
: Let A be a given point whose position vector $\vec{a}$ and $\vec{b}$ be given vector.

Let 'l' be the line passing through the point A and is parallel to a given vector $\vec{b}$.

Let 'P' be any point on the line 'l' whose position vector is $\vec{r}$.

Then, $\overline{AP}$ is parallel to the vector $\vec{b}$.

$\overline{AP}=\lambda\vec{b}$, where λ is a real number.



$\Rightarrow\ \overline{OP}-\overline{OA}=\lambda\vec{b}$

$\vec{r}-\vec{a}=\lambda\vec{b}$

$\therefore\ \vec{r}=\vec{a}+\lambda\vec{b}$

which is the required vector equation of the line.

Let (x_1, y_1, z_1) be the co-ordinates of the point A. Let a, b, c be the direction ratios of the given line $\vec{b}$. Let $P(x, y, z)$ be a point on the required line.

$\therefore\ \vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$

and $\vec{b}=a\hat{i}+b\hat{j}+c\hat{k}$

Substitute these values in $\vec{r}=\vec{a}+\lambda\vec{b}$ and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$x=x_1+\lambda a \Rightarrow x-x_1=\lambda a$

$y=y_1+\lambda b \Rightarrow y-y_1=\lambda b$

$z=z_1+\lambda c \Rightarrow z-z_1=\lambda c$

Eliminating the parameter λ, we get

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

is the required equation of given line.

48. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize

a) Exactly once **b) at least once?**

Here $n=50,\ p=\frac{1}{100},\ q=\frac{99}{100}$

$$P(X=x)={}^{n}C_x(q)^{n-x}(p)^x={}^{50}C_x\left(\frac{99}{100}\right)^{50-x}\cdot\left(\frac{1}{100}\right)^x$$

Required:

(b) $P(X\geq 1)=1-P(0)=1-\left(\frac{99}{100}\right)^{50}$

(a) $P(X=1)=\frac{1}{2}\cdot\left(\frac{99}{100}\right)^{49}$

PART-E

III. Answer any one of the following questions.

49. a) Prove that $\int_{-a}^{a} f(x)dx=\begin{cases} 2\int_0^a f(x)dx, \text{if } f(x) \text{ is an even function} \\ 0, \text{if } f(x) \text{ is an odd function}\end{cases}$

and hence evaluate $\int_{-1}^{1} \sin^5 x \cos^4 x\, dx$

Ans: We have

$$\int_{-a}^{a} f(x)\,dx=\int_{-a}^{0} f(x)\,dx+\int_{0}^{a} f(x)\,dx \quad \ldots(1)$$

Consider $\int_{-a}^{0} f(x)\,dx$

Put $x=-t \Rightarrow dx=-dt$

When $x=-a,\ t=a$

When $x=0,\ t=0$

$$\int_{-a}^{0} f(x)\,dx=\int_{a}^{0} f(-t)\,(-dt)=\int_{0}^{a} f(-t)\,dt$$

$$=\int_{0}^{a} f(-x)\,dx$$

$\therefore$ (1) reduces to

$$\int_{-a}^{a} f(x)\,dx=\int_{0}^{a} f(-x)\,dx+\int_{0}^{a} f(x)\,dx$$

$$=\int_{0}^{a}[f(-x)+f(x)]\,dx \qquad \ldots(2)$$

If 'f' is an even function i.e., $f(-x)=f(x)$,

then (2) reduces to

$$\int_{-a}^{a} f(x)\,dx=\int_{0}^{a}[f(x)+f(x)]\,dx$$

$$= 2\int_0^a f(x)\, dx$$

If 'f' is an odd function i.e., $f(-x) = -f(x)$,

then (2) reduces to

$$\int_{-a}^{a} f(x)\, dx = \int_0^a [-f(x) + f(x)]\, dx = 0$$

b) show that $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x^2)$

$$\text{LHS} = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$\begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix} \because C_1 \to C_1 + C_2 + C_3$$

Take out $(5x+4)$ from C_1

$$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & 4-x & 0 \\ 0 & 0 & 4^{-x} \end{vmatrix}$$

$\because R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

Expanding along R_1, we get

$$\text{LHS} = (5x+4)[1(4-x)(4-x) - 0 + 0]$$

$$(5x+4)(4-x)^2 = \text{RHS}$$

50. **a) Maximise $z = 4x + y$**

Subject to constraint:

$x + y \leq 50$

$3x + y \leq 90$

$x \geq 0$

$y \geq 0$

by graphical method

$x+y \leq 50$ $\quad$ $3x+y \leq 90$

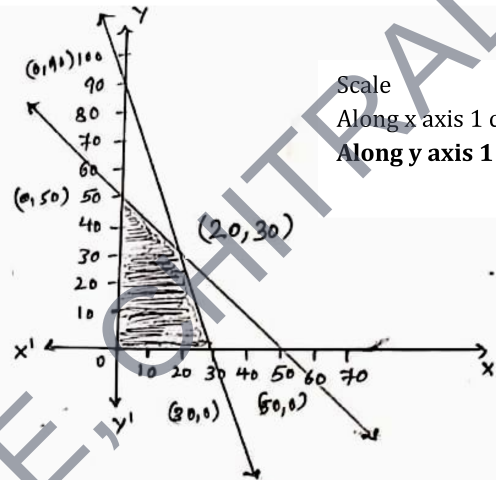
$x+y = 50$ $\quad$ $3x+y = 90$

x	0	50
y	50	0

(0,50) (50,0)

x	0	30
y	90	0

(0,90), (30,0)



Scale
Along x axis 1 cm= 10 units
Along y axis 1 cm= 10 units

Shaded region is the Feasible region which is bounded and bounded by the corner points (0,0), (0,50), (20,30), (30,0)

Corner points	$Z = 4x + y$
(0,0)	0
(0,50)	50
(20,30)	110
(30,0)	120

Maximum value of $Z = 120$ at $(30,0)$

b) Find the value of K, if $f(x) = \begin{cases} Kx + 1, if x \leq \pi \\ cosx, if x > \pi \end{cases}$ ***is continuous at*** $x = \pi$

$f(\pi) = K\pi + 1$

$LHL = \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} Kx + 1 = K\pi + 1$

$RHL = \lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} \cos x = \cos\pi = -1$

Given f(x) is continuous at x=π

$LHL = RHL$

$K\pi + 1 = -1$

$K\pi = -2$

$\boxed{K = \frac{-2}{\pi}}$